Discrete Dynamics for Behavioral Bubbles

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Introduction

In this talk we focus our attention on the relations between a rational model of bubbles, defined as classic, and a behavioral model, called representative.

Giglio et al. (2016) demonstrated that, as for the classic rational models, bubbly dynamics are not detected looking for a failure of the transversality condition, although advanced econometric tests such as (Phillips et al. 2015) measured positive bubbly dynamics in the same assets and during the same time spans.
Introduction

The model of representative bubbles, a behavioral model, arises from this gap in order to face up with the following statement:

Giglio et al. 2016

In this light, the fact that our tests rule out the presence of classic rational bubbles in housing markets might shift the literature toward focusing on other, more empirically plausible models of bubbles. The implications of including such possibly irrational bubbles in macroeconomic models remain an interesting avenue for further research.
Introduction

Our model follows that of Giglio et al. (2016), who tested whether a classic rational bubble model can depict bubbly episodes.

This type of rational model detects only those bubbles that can arise from an intertemporal inconsistency on the expected equilibrium of prices; in a few words: if the transversality condition on the bubbly component fails to hold.
Introduction

In Diba & Grossmann it has been shown that a rational bubble rises only if it is already present at the first trading of a bubbly asset, that is: if a bubble does not blow at the first trade, it cannot rise at any subsequent time.

Maggiori et al. tested for the existence of rational bubbles in the housing market that preceded the turmoil of 2007: they found that there has been not a failure of the transversality condition and then rational bubbles can be ruled out as a culprit for overvalued houses.

Strati defined a behavioral model of bubble that is able to consider over-optimism (resp. over-pessimism) in the market by taking into account a behavioral component that is neglected in the classic rational case, that can detect a growing bubble even in cases in which the transversality condition is fulfilled.
**Introduction**

A key component is that a bubble may arise because of an exogenous flow of positive news that overheats the perception of a so-called *local thinker* who is excessively sensitive to new information.

We aim at clarify the formal inception of this kind of behavioral bubble driven by the heuristic of representativeness: the *representative bubble*.

We will find that a representative bubble may blow at any time, hence it is at odds with the inception of a rational bubble defined as defined in Dida & Grossmann.

These positive shocks, or displacements, inflate the probability of an expected growth state: it is a diagnostic signal of their representativeness, thus the expectations that follow from this heuristic are called *diagnostic*. 
The representative expected state inflates with respect to the rational expectations component, that is to say, diagnostic expectations exaggerate the direction of the path followed by the rational ones without changing the direction: diagnostic expectations are based upon a *kernel of truth*.
Introduction

Representative bubbles show up from the conjoined presence of psychological factors and limits of arbitrage.

A summary of the differences, which also appear in a general model, is as follows:

1. **Inception**: Classic rational bubbles arise from the first trade by definition of the efficient market hypothesis, while representative bubbles may appear at any time because of psychological factors instigated by exogenous shocks (displacements);

2. **Causes**: Classic rational bubbles arise from pure speculation given the failure of the transversality condition, while representative bubbles pop up from a biased expectations formation by which agents fail to understand a signal rationally.
We shall define the condition by which representative bubbles can inflate at any time: their inception is linked to the level of rationality of the agents. From Diba & Grossmann, it will be clear that for fully rational agents, if a rational bubble does not exist at \( t \geq 0 \), then that rational bubble can get started neither at \( t + 1 \) nor at any subsequent time. However, this result is not true when agents are *local thinkers*, that is those agents whose expectations are driven by the heuristic of representativeness.
The rational case

Following Diba & Grossmann, we define a first order stochastic difference equation in $y$ with a forcing variable $x$ and a set of information $I_t$, a set containing past realizations of both a exogenous and endogenous variables up to current time $t$.

We describe a rational asset-price bubble in the context of (linear) rational expectations $\mathbb{E}_t[\cdot]$,\(^1\) that is

$$\alpha \mathbb{E}_t[y_{t+1} | I_t] = y_t - x_t. \quad (1)$$

\(^1\)The rational expectations of a variable of a model formed by an individual at time $t$ are defined as the mathematical expectation conditional on the information set $I_t$. 
The rational case

For the eigenvalue $\alpha > 1$, equation (1) is convergent, it can be solved by the forward looking method. In particular, by applying the law of iterated expectations we obtain

$$y_t = \alpha \mathbb{E}_t[y_{t+1} | l_t] + x_t$$

$$= \alpha^2 \mathbb{E}_t[y_{t+2} | l_t] + \alpha \mathbb{E}_t[x_{t+1} | l_t] + x_t$$

$$\ldots$$

$$= \alpha^{T+1} \mathbb{E}_t[y_{t+1+T} | l_t] + \sum_{i=0}^{T} \alpha^i \mathbb{E}_t[x_{t+i} | l_t],$$

(2)

taking into account that if $T = 0$, then the expectation at $T = 0$ coincides with $x_t$ and $l_t = l$ for every $t \in \{0, \ldots, T\}$. 
The rational case

The forward looking solution, or *fundamental solution*, is the second term of the right hand side of (2) and it is usually denoted by $F_t$, that is

$$F_t = \sum_{i=0}^{T} \alpha^i \mathbb{E}_t[x_{t+i} \mid l_t]. \quad (3)$$

The first term of the right hand side of (2) is called *bubble component* $B_t$ and from which it is obtained the *bubble solution*. In order to have a unique solution of (1), it should be imposed the transversality condition on the bubble component

$$\lim_{T \to \infty} \alpha^{T+1} \mathbb{E}_t[y_{t+1+T} \mid l_t] = 0, \quad (4)$$

otherwise $y_t = F_t$, would have been only one of the possible solutions of

$$y_t = F_t + B_t.$$
The rational case

Diba & Grossmann found a convincing result: a bubble cannot blow at $t+i$ if it is not already present at $t$, at the inception.

From this fact, the bubble component can be seen as an innovation comprising new information available at $t+1$. The information encompassed in those changes of $x_t$ can affect $y_t$ directly or by modifying the information set $I_t = \{x_t, x_{t-1}, \ldots\}$, hence affecting $y_t$ indirectly by the expectation term in (1).

The bubble component may be not connected with the fundamental information and since the expected future values of $z_{t+i}$ are zero by definition of the rational model, then the new information of innovation seems not to be of any help in the assessment of a positive $B_{t+1}$. 
The rational case

From equation (5), it follows that $\mathbb{E}_t[B_{t+j} \mid I_t] = \alpha^{-j}B_t$, there is an explosive conditional expectation on the value of the bubble component for $B_t > 0$, and $\mathbb{E}_t[B_{t+j}] = \infty$ for $j \to \infty$ and $1/\alpha > 1$.

If $y_t$ is a (asset) price, then by the free disposal of the Walrasian equilibrium, $y_t \geq 0$ the possibility of a negative rational bubble is ruled out.

If the free disposal implies a nonnegativity for $y_t$, then the bubble component of $y_{t+1}$ must satisfy $B_{t+1} \geq 0$.

From equation (6) and the nonnegativity condition we have:

$$z_{t+1} \geq -\lambda B_t.$$  (7)
The rational case

By equation (6) the bubble component is equal to the innovation brought about by $z_{t+1}$, hence if one is zero, the other must be zero: if a bubble exists, it must exist from the inception.

Another property follows from the transversality condition of equation (4), that is: if there is a departure from the optimal path of $y_{t+i}$ and thus a failure of the transversality condition (4), then from the rational expectations equilibrium, arbitrageurs adjust the inflated value of $y_{t+i}$ at once, using up the positive bubble component.
The Concept of Representativeness

Representativeness is an assessment of the degree of the correspondence between a sample and a population... more generally, between an outcome and a model.

Tversky and Kahneman (1983)

A person who follows this heuristic evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated.

Kahneman and Tversky (1972)
The Concept of Representativeness

For a distribution of a trait $T$ in a group $\Gamma$, a decision maker (DM) will assess the *relative* frequency of a particular trait $T = t$ in $\Gamma$ with respect to the frequency of the same trait $T = t$ in a relevant comparison group $\Gamma^c$. The true distribution is $\Pr(T = t | \Gamma)$.

The heuristic of representativeness $\mathcal{R}(\cdot)$ has been formalized in mGennaioli and Shleifer as follows

$$\frac{\Pr(T = t | \Gamma)}{\Pr(T = t | \Gamma^c)}.$$

It is plain that representativeness defines the inflation of the likelihood of traits whose *objective* probability rises the most in $\Gamma$ relative to the reference context $\Gamma^c$ (Bordalo et al.).
Diagnostic Expectations

There is a discrete time $t = 1, 2, \ldots$, and two states of the world: growth $g$ and recession $r$.

At any time there is a signal $s \in \mathbb{R}_+$ about the next state space, with a growth signal $\bar{s}$ and a recession signal $\underline{s}$ with $\underline{s} < \bar{s}$.

The signals are characterized in the following way:
$\Pr(\underline{s} \mid r) = \gamma$, $\Pr(\underline{s} \mid g) = 1 - \beta$ for which $\gamma > \beta \geq 1/2$: a bad signal $\underline{s}$ reduces the probability of expecting a growth rate and it is a very strong signal for a looming recession.
Assumption 1: There is a prior probability $\pi_k$ with $k = \{g, r\}$ for which $\pi_g > \pi_r$.

Denote a generic state space $\Omega$ in which $\omega$ is a realization. In particular, the state of the economy is a random variable $\Omega_t$. Define by $(\Omega_{t+1} = \omega_{t+1})$ the realization of the state $\omega_{t+1}$ at $t+1$. If $(\Omega_{t+1} = \omega_{t+1} \mid \Omega_t = \omega_t)$ then it is intended that the realization $\omega_{t+1}$ at $t+1$ hinges on the occurrence of the state $\omega_t$ at $t$. Now suppose that there exists a smooth density function $f(\cdot)$ for which in a Bayesian framework

$$F(s \mid \omega_{t+1}) = \int_{s}^{+\infty} f(\Omega_{t+1} = \omega_{t+1} \mid \Omega_t = \omega_t)d\omega_{t+1} \quad (8)$$

is the rational cumulative distribution of the probability that the state $\omega_{t+1}$ is a growth state.
Assumption 1: There is a prior probability $\pi_k$ with $k = \{g, r\}$ for which $\pi_g > \pi_r$.

Define $\Gamma \equiv \{\Omega_t = \omega_t\}$ as the group that depicts all the possible future states $\omega_{t+1}$ whose values depend on the current state $\Omega_t = \omega_t$. Moreover, define its comparison group $\Gamma^c \equiv \Omega \setminus \Gamma$. The comparison group may be defined as the rational expectation formed at $t - 1$ for $\omega_{t+1}$ in which no new pieces of information occur, that is $\Gamma^c \equiv \{\Omega_t = \mathbb{E}_{t-1}(\omega_t)\}$ (Bordalo et al.).

**Definition**

The representativeness of a state $\omega_t$ at $t$ for a group $\Gamma$ is defined as

$$
R(\omega_t, \Gamma, \Gamma^c) = \frac{f(\Omega_{t+1} = \omega_{t+1} \mid \Omega_t = \omega_t)}{f(\Omega_{t+1} = \omega_{t+1} \mid \Omega_t = \mathbb{E}_{t-1}(\omega_t))}.
$$

(9)
Getting Formal

The representativeness is set up by considering a mental process that formalizes the similarity between group $\Gamma \equiv \{ \Omega_t = \omega_t \}$ and the comparison group, by considering a limited and selective retrieval from memory: how scenarios become accessible from memory. They come to mind following an order dictated by their representativeness rather than by their probability. This is formalized as follows

$$d(\omega_1) \equiv f(\Omega_1 = \omega_1 \mid \Omega_0 = \omega_0) \times \left[ \frac{f(\Omega_1 = \omega_1 \mid \Omega_0 = \omega_0)}{f(\Omega_1 = \omega_1 \mid \Omega_0 = \mathbb{E}_1(\omega_0))} \right]^{\theta \frac{1}{Z_e}}$$  

- If $\theta = 0 \Rightarrow$ Rational Agents
- If $\theta > 0 \Rightarrow$ Local Thinking
The Representative Case

A rational bubble is present from the inception of a trade, and when it bursts, it cannot arise again in the same asset. Moreover, a bubble component can be positive just for a very short span of time. From the bubble defined in Strati, these conclusions should be revised.

Define equation (1) in the following way

\[
y_t = \alpha E^d_t [y_{t+1}] + \chi_t
\]  

(11)

in which \( E^d_t [\cdot] \) is the diagnostic expectation from which the representative density function (10) is explicitly defined as a normal distribution

\[
d(y_{t+1}) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(y_{t+1} - E_t(y_{t+1}))^2}{2\sigma^2} \right) \times \frac{\exp \left( -\frac{(y_{t+1} - E_t(y_{t+1}))^2}{2\sigma^2} \right)}{\theta Z} \theta Z
\]

(12)
The Representative Case

Thus we obtain

$$d(y_{t+1}) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} [(1+\theta)(y_{t+1} - \mathbb{E}_t(y_{t+1}))^2 - \theta(y_{t+1} - \mathbb{E}_{t-1}(y_{t+1}))^2] \right) \frac{1}{Z}.$$ 

Moreover, since

$$[(1 + \theta)(y_{t+1} - \mathbb{E}_t(y_{t+1}))^2 - \theta(y_{t+1} - \mathbb{E}_{t-1}(y_{t+1}))^2] = (y_{t+1} - \mathbb{E}_t(y_{t+1}) + \theta[\mathbb{E}_t(y_{t+1}) - \mathbb{E}_{t-1}(y_{t+1})]^2 = (y_{t+1} - \mathbb{E}_t^d(y_{t+1}))^2,$$

we rewrite equation (12) as

$$d(y_{t+1}) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} [(y_{t+1} - \mathbb{E}_t^d(y_{t+1}))^2] \right) \frac{1}{Z}.$$
The Representative Case

In this linear environment it is plain how the heuristic of representativeness causes an exaggeration in the judgment based on rational expectations, that is

\[
\mathbb{E}_t^d(y_{t+1}) \equiv \mathbb{E}_t(y_{t+1}) + \theta[\mathbb{E}_t(y_{t+1}) - \mathbb{E}_{t-1}(y_{t+1})], \quad (13)
\]

in which \(\theta\) shifts the (gaussian) rational distribution to the right. Since the rational (gaussian) distribution remains the feasible one for the market, then this shift causes a neglection of the tail risk encompassed in the (gaussian) rational distribution and over-optimism: good news cause neglect of downside risks and an increasing optimism.
Strati has been shown that for positive shocks to fundamentals, the so called *displacements* (see kindle), there is an excess of volatility which brings about a positive forecast error that, since it is not orthogonal to the information available at the time of the forecast, it triggers the presence of a positive component directly into the fundamental solution. This result tells us that there is no need to a positive bubble component: a representative bubble (driven by representativeness) overvaluation anyway. Let us see this reasoning formally.
Diagnostic Expectations: Extrapolation and Neglection

The excess volatility is justified by the fact that rational expectations are distorted towards the direction of realized news. Positive news can cause over-optimistic expectations (vice versa for negative news).

Consider the cumulative distribution

\[
F_t^d(s) = \int_s^{+\infty} d(y_{t+1})dy_{t+1},
\]

that describes the probability that positive news continue to hit the model. Note that for a contingent current value \(y_t\)

\[
\frac{\partial \ln F_t^d(s)}{\partial y_t} = \frac{1}{F_t^d(s)} \frac{\partial F_t^d(s)}{\partial \mathbb{E}_t^d(y_{t+1})} \cdot \frac{\partial \mathbb{E}_t^d(y_{t+1})}{\partial y_t}.
\]
Diagnostic Expectations: Extrapolation and Neglection

i. the first term on the right hand side is

\[
\frac{1}{F_t^d(s)} \frac{\partial F_t^d(s)}{\partial \mathbb{E}_t^d(y_{t+1})} = \int_\bar{s}^\infty \frac{y_{t+1} - \mathbb{E}_t^d(y_{t+1})}{\sigma^2} \cdot \exp \left[ - \frac{(y_{t+1} - \mathbb{E}_t^d(y_{t+1}))^2}{2\sigma^2} \right] \frac{dy_{t+1}}{F_t^d(s)} =
\]

\[
\frac{1}{\sigma^2} \int_\bar{s}^\infty y_{t+1} \exp \left( - \frac{1}{2\sigma^2} \left[ (y_{t+1} - \mathbb{E}_t^d(y_{t+1}))^2 \right] \right) \frac{dy_{t+1}}{F_t^d(s)} - \frac{\mathbb{E}_t^d(y_{t+1})}{\sigma^2}
\]

that is to say

\[
\frac{\partial \ln F_t^d(s)}{\partial y_t} = \left[ \mathbb{E}_t^d(y_{t+1} \mid y_{t+1} \geq \bar{s}) - \mathbb{E}_t^d(y_{t+1}) \right] \frac{1}{\sigma^2};
\]
Diagnostic Expectations: Extrapolation and Neglection

\( ii) \) as for the second term, assume normal densities and AR(1) process for \( E_t(y_{t+1}) = a + by_t \), rewriting (13) in these terms, then

\[
\frac{\partial E_t^d(y_{t+1})}{\partial y_t} = b(1 + \theta) > 0.
\] (15)

It follows that

\[
\frac{\partial \ln F_t^d(s)}{\partial y_t} > 0.
\] (16)
From equations (16) and (15), an over-optimistic behaviour is driven by $\theta$ that triggers an excess of volatility detected by the variance

$$\text{Var}\left[\mathbb{E}_t^d(y_{t+1})\right] = \text{Var}\left[(1 + \theta)\mathbb{E}_t(y_{t+1}) - \mathbb{E}_{t-1}(y_{t+1})\right]$$

$$= (1 + \theta)^2 \text{Var}\left[\mathbb{E}_t(y_{t+1})\right].$$

(17)
Diagnostic Expectations: Extrapolation and Neglection

That in turn, taking into account equation (13), sets off a positive predictable error

\[
E_t[y_{t+1} - E_t^d(y_{t+1})] = \\
E_t\left(y_{t+1} - (E_t(y_{t+1}) + \theta[E_t(y_{t+1}) - E_{t-1}(y_{t+1})])\right) \\
= E_t(y_{t+1}) - E_t(E_t(y_{t+1})) + E_t(\theta E_t(y_{t+1})) - E_t(\theta(E_{t-1}(y_{t+1})) \\
= -\theta[E_t(y_{t+1}) - E_{t-1}(y_{t+1})]. \quad (18)
\]

Since past values of \( y \) affect the formation of the current expectations, then the orthogonality condition is violated since the extrapolation of past good news. Moreover, the negative term of the error is justified by the fact that an ex-ante overvaluation is symptomatic of a future fall of the value.
The Representative Bubble

As for the rational case, let us find a forward solution (now) for equation (11), that is

\[ y_t = \alpha^{T+1} \mathbb{E}_{t-1+T} \left[ y_{t+1+T} \mid l_t \right] + \sum_{i=0}^{T} \alpha^i \mathbb{E}_{t+1} \left[ x_{t+i} \mid l_t \right]. \]

Since we are interested in the case in which a bubble exists also in case of a zero bubble component, consider that the transversality condition is fulfilled, and thus, even for a \( \theta > 0 \), equation (13) tends to zero since the free disposal on \( t - 1 \).
Assumption $\lim_{T \to \infty} \mathbb{E}_t[\alpha_{t,T-1} \rho_{t,T-1}] = 0$

From this assumption, it follows that

$$y_t = \sum_{i=0}^{T} \alpha^i \mathbb{E}_{t+1}^{d}[x_{t+i} \mid l_t]. \quad (19)$$

Differently from equation (3), in equation (19) expectations are assumed to be diagnostic. In particular, agents are over-sensitive to positive news that hit fundamentals (displacements). This biased sensitivity causes an overconfidence among agents that in turn triggers both: a neglection of downside risks and a positive predictable error since the extrapolative nature of these expectations.
Assumption $\lim_{T \to \infty} \mathbb{E}_t[\alpha_{t,T-1}\rho_{t,T-1}] = 0$

By implicitly assuming that the feasible path is that which follows rational expectations, the difference between the path driven by representativeness and that of the rational one, suggests that for equation 19 the reasoning behind the forward solution has to be different. It is now clear that the fundamental value of equation (19) is biased by a positive error. This error, as we stated, derives from the excess of volatility of (17); that is

$$\epsilon^d_t = [\mathbb{E}^d_t(x_{t+1}) - \mathbb{E}_t(x_{t+1})]$$

that, from equation (13), can be rewritten as

$$\epsilon^d_t \equiv \theta[\mathbb{E}_t(x_{t+1}) - \mathbb{E}_{t-1}(x_{t+1})].$$

where $\epsilon^d_t$ is the error in expectations caused by diagnostic expectations.
Assumption \( \lim_{T \to \infty} \mathbb{E}_t[\alpha_{t,t+T-1} \rho_{t+T-1}] = 0 \)

The solution of Equation (19) is thus

\[
y_t = \sum_{i=0}^{T} \alpha^i \mathbb{E}_{t+1}[x_{t+i} \mid I_t] + \epsilon^d_t \tag{21}
\]

with \( \epsilon^d_t > 0 \), that is \( y_t = F_t + \epsilon^d_t \). In Strati it has been shown that for a homogeneous euphoria of the market, that can be not detected by rational bubbles (since the transversality condition on \( B_t \)), there can exist a bubble driven by representativeness.
When does it start?

Differently from the rational case, if a bubble is representative, that bubble can start (resp. burst) at any time, depending on both the persistence and sensitivity of the agents’s expectations to positive (resp. negative) shocks.

The level of rationality of the agents is thus crucial for the inception of a bubble that does not always coincide with the first trade of an asset.

Moreover, the parameter $\theta$ can either increase or decrease with respect to the institutions in which agents operate, or with respect to how and what comes to mind when they make a decision.
When does it start?

**Proposition**

A bubble driven by representativeness can arise in the related environment at any \( t \geq 0 \).

**Remark**

The representative bubble is caused by a limited rationality of the agents for every credible shock that hits agents’s beliefs and that can occur at any \( t \geq 0 \).
A Comparison

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Conclusions

We have defined how representative bubbles arise and where they differ from the classic rational models.

This is the first self-contained discussion on representative bubbles, which is of utmost importance for making their precise meaning clear.

We showed that representative bubbles may blow at any time and with no restrictions.
Conclusions

The implications are thus the followings:

- From proposition 37, a representative bubble can start at any time due to *displacements* that follow from extrapolative expectations and neglect of tail risks. In short, from diagnostic expectations;

- as rational bubbles, the representative dynamics cannot be negative since the free disposal;

- following proposition 37, representative bubbles can arise, burst at zero and then grow again in the same asset;

- a representative bubble is not dependent on the failure of the transversality condition, hence it can be persistent and distorted.
References


References

References


Thank You!!